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Parameters of Lorentz Matrices and Transitivity in Polarization Optics

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In the context of applying the Lorentz group theory to polarization optics in the frames of Stokes–Mueller formalism, some properties of the Lorentz group are investigated. We start with the factorized form of arbitrary Lorentz matrix as a product of two commuting and conjugate 4×4 -matrices, $L(q, q) = A(q_a)A^*(q_a)$; $a = 0, 1, 2, 3$. Mueller matrices of the Lorentzian type $M = L$ are pointed out as a special sub-class in the total set of 4×4 matrices of the linear group $GL(4, R)$. Any arbitrary Lorentz matrix is presented as a linear combination of 16 elements of the Dirac basis. On this ground, a method to construct parameters q_a by an explicitly given Lorentz matrix L is elaborated. It is shown that the factorized form of $L=M$ matrices provides us with a number of simple transitivity equations relating couples of initial and final 4-vectors, which are defined in terms of parameters q_a of the Lorentz group. Some of these transitivity relations can be interpreted within polarization optics and can be applied to the group-theoretic analysis of the problem of measuring Mueller matrices in optical experiments.

I. ON ESTABLISHING THE PARAMETERS OF LORENTZ MATRICES FROM THEIR EXPLICIT FORM

In the context of applying the Lorentz group theory to polarization optics in the frames of Stokes–Mueller formalism, some properties of the Lorentz group are investigated (see [3]–[9]; the notation according to [2, 9] is used). We start with a factorized representation for Lorentz matrices[11]

$$[L_b^a(q, \bar{q}^*)] = A(q)A^*(q), \quad (1)$$

where

$$A(q) = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ -q_1 & q_0 & -iq_3 & iq_2 \\ -q_2 & iq_3 & q_0 & -iq_1 \\ -q_3 & -iq_2 & iq_1 & q_0 \end{pmatrix}, \quad A^*(q) = \begin{pmatrix} q_0^* & -q_1^* & -q_2^* & -q_3^* \\ -q_1^* & q_0^* & iq_3^* & -iq_2^* \\ -q_2^* & -iq_3^* & q_0^* & iq_1^* \\ -q_3^* & iq_2^* & -iq_1^* & q_0^* \end{pmatrix}. \quad (2)$$

We shall prove that one can easily construct matrices for which the complex vectors q and q^* are eigenvectors. Indeed, the following identities hold[12]

$$L q = \bar{q}^* , \quad L q^* = \bar{q} , \quad \bar{q}^* = (q_0^*, -\mathbf{q}^*) , \quad \bar{q} = (q_0, -\mathbf{q}) . \quad (3)$$

With the use of the special element

$$\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

the relations from (3) read

$$\delta L q = q^* , \quad \delta L q^* = q . \quad (4)$$

whence, through an elementary process we get

$$\delta L \delta L q = \delta L q^* = q , \quad \delta L \delta L q^* = \delta L q = q^* .$$

i.e.,

$$Q = (\delta L)^2 , \quad Qq = q , \quad Qq^* = q^* . \quad (5)$$

Allowing the pseudo-orthogonality of Lorentz transformations[13], i.e., $\delta L \delta = (L^t)^{-1}$, the matrix Q may be represented as

$$Q = (L^t)^{-1} L . \quad (6)$$

In particular, for the orthogonal subgroup of rotations, we have

$$L = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} , \quad (L^t) = L^{-1} , \quad Q = L^2 = \begin{pmatrix} 1 & 0 \\ 0 & O^2 \end{pmatrix} , \quad (7)$$

with $OO^t = I_3$, $\det O = 1$. The explicit form for an arbitrary Euclidean rotation is fixed by the parameters $q_0 = n_0$, $q_j = -in_j$, so that

$$L = \begin{pmatrix} n_0 & in_1 & in_2 & in_3 \\ in_1 & n_0 & -n_3 & n_2 \\ in_2 & n_3 & n_0 & -n_1 \\ in_3 & -n_2 & n_1 & n_0 \end{pmatrix} \begin{pmatrix} n_0 & -in_1 & -in_2 & -in_3 \\ -in_1 & n_0 & -n_3 & n_2 \\ -in_2 & n_3 & n_0 & -n_1 \\ -in_3 & -n_2 & n_1 & n_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} ,$$

where

$$O = \begin{pmatrix} 1 - 2(n_2^2 + n_3^2) & -2n_0n_3 + 2n_1n_2 & +2n_0n_2 + 2n_1n_3 \\ +2n_0n_3 + 2n_1n_2 & 1 - 2(n_3^2 + n_1^2) & -2n_0n_1 + 2n_2n_3 \\ -2n_0n_2 + 2n_1n_3 & +2n_0n_1 + 2n_2n_3 & 1 - 2(n_1^2 + n_2^2) \end{pmatrix}. \quad (8)$$

The 4×4 -matrix L acts on the 4-vector n_a according to

$$\begin{pmatrix} 1 & 0 \\ 0 & O \end{pmatrix} \begin{pmatrix} n_0 \\ -i\mathbf{n} \end{pmatrix} = \begin{pmatrix} n_0 \\ -i\mathbf{n} \end{pmatrix}. \quad (9)$$

Now let us consider arbitrary pseudo-Euclidean rotations – these are fixed by the parameters $q_0 = m_0$, $q_j = m_j$, $m_0^2 - \mathbf{m}^2 = +1$.

By denoting $\mathbf{e} = (e_1, e_2, e_3)$, we have

$$m_0 = \cosh(\chi/2), \quad m_j = \sinh(\chi/2) e_j, \quad \mathbf{e}^2 = 1; \quad (10)$$

we get the following representation

$$L = \begin{pmatrix} \cosh \chi & -\sinh \chi e_1 & -\sinh \chi e_2 & -\sinh \chi e_3 \\ -\sinh \chi e_1 & 1 + (\cosh \chi - 1)e_1^2 & (\cosh \chi - 1)e_1 e_2 & (\cosh \chi - 1)e_1 e_3 \\ -\sinh \chi e_2 & (\cosh \chi - 1)e_1 e_2 & 1 + (\cosh \chi - 1)e_2^2 & (\cosh \chi - 1)e_2 e_3 \\ -\sinh \chi e_3 & (\cosh \chi - 1)e_1 e_3 & (\cosh \chi - 1)e_2 e_3 & 1 + (\cosh \chi - 1)e_3^2 \end{pmatrix}. \quad (11)$$

Then, we readily find by direct calculation that

$$Q = (\delta L)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

This identity can be proved in a different way. Since the pure Lorentz transformation is given by some symmetrical matrix, the relation (6) takes the form

$$Q = (L^t)^{-1} L = L^{-1} L = I, \quad (13)$$

which coincides with (12).

We shall further explicitly describe the case when the arbitrary Lorentz matrix is given by blocks

$$L = \begin{pmatrix} K & M \\ N & L \end{pmatrix},$$

block (K)

$$\begin{aligned} L_{00} &= (q_0 q_0^* + q_1 q_1^*) + (q_2 q_2^* + q_3 q_3^*), & L_{01} &= -(q_0 q_1^* + q_1 q_0^*) + i(q_2 q_3^* - q_3 q_2^*), \\ L_{10} &= -(q_0 q_1^* + q_1 q_0^*) - i(q_2 q_3^* - q_3 q_2^*), & L_{11} &= (q_0 q_0^* + q_1 q_1^*) - (q_2 q_2^* + q_3 q_3^*); \end{aligned}$$

block (M)

$$\begin{aligned} L_{22} &= (q_0 q_0^* - q_1 q_1^*) + (q_2 q_2^* - q_3 q_3^*), & L_{23} &= i(q_0 q_1^* - q_1 q_0^*) + (q_2 q_3^* + q_3 q_2^*), \\ L_{32} &= -i(q_0 q_1^* - q_1 q_0^*) + (q_2 q_3^* + q_3 q_2^*), & L_{33} &= (q_0 q_0^* - q_1 q_1^*) - (q_2 q_2^* - q_3 q_3^*); \end{aligned}$$

block (N)

$$\begin{aligned} L_{02} &= -(q_0 q_2^* + q_2 q_0^*) - i(q_1 q_3^* - q_3 q_1^*), & L_{03} &= -(q_0 q_3^* + q_3 q_0^*) + i(q_1 q_2^* - q_2 q_1^*), \\ L_{12} &= i(q_0 q_3^* - q_3 q_0^*) + (q_1 q_2^* + q_2 q_1^*), & L_{13} &= -i(q_0 q_2^* - q_2 q_0^*) + (q_1 q_3^* + q_3 q_1^*); \end{aligned}$$

block (L)

$$\begin{aligned} L_{20} &= -(q_0 q_2^* + q_2 q_0^*) + i(q_1 q_3^* - q_3 q_1^*), & L_{21} &= -i(q_0 q_3^* - q_3 q_0^*) + (q_1 q_2^* + q_2 q_1^*), \\ L_{30} &= -(q_0 q_3^* + q_3 q_0^*) - i(q_1 q_2^* - q_2 q_1^*), & L_{31} &= i(q_0 q_2^* - q_2 q_0^*) + (q_1 q_3^* + q_3 q_1^*). \end{aligned}$$

A definite ordering structure will be seen in the matrix L , if one decomposes L into symmetric S and skew-symmetric A parts:[14]

$$\begin{aligned} S_{00} &= L_{00} = q_0 q_0^* + q_1 q_1^* + q_2 q_2^* + q_3 q_3^*, & S_{11} &= L_{11} = q_0 q_0^* + q_1 q_1^* - q_2 q_2^* - q_3 q_3^*, \\ S_{22} &= q_0 q_0^* + q_2 q_2^* - q_1 q_1^* - q_3 q_3^*, & S_{33} &= q_0 q_0^* + q_3 q_3^* - q_1 q_1^* - q_2 q_2^*, \\ S_{01} &= -(q_0 q_1^* + q_1 q_0^*), & S_{02} &= -(q_0 q_2^* + q_2 q_0^*), & S_{03} &= -(q_0 q_3^* + q_3 q_0^*), \\ S_{12} &= q_1 q_2^* + q_2 q_1^*, & S_{13} &= q_1 q_3^* + q_3 q_1^*, & S_{23} &= q_2 q_3^* + q_3 q_2^*, \\ A_{01} &= i(q_2 q_3^* - q_3 q_2^*), & A_{02} &= i(q_3 q_1^* - q_1 q_3^*), & A_{03} &= i(q_1 q_2^* - q_2 q_1^*), \\ A_{12} &= -i(q_3 q_0^* - q_3^* q_0), & A_{13} &= +i(q_2 q_0^* - q_2^* q_0), & A_{23} &= -i(q_1 q_0^* - q_1^* q_0). \end{aligned}$$

We easily get

$$\begin{aligned} Q_0^2 &= k_0 k_0^* = \frac{1}{4} \text{Sp } L = \frac{S_{00} + S_{11} + S_{22} + S_{33}}{4}, \\ Q_1^2 &= q_1 q_1^* = \frac{S_{00} + S_{11}}{2} - \frac{1}{4} \text{Sp } L = \frac{S_{00} + S_{11} - S_{22} - S_{33}}{4}, \\ Q_2^2 &= q_2 q_2^* = \frac{S_{00} + S_{22}}{2} - \frac{1}{4} \text{Sp } L = \frac{S_{00} + S_{22} - S_{11} - S_{33}}{4}, \\ Q_3^2 &= q_3 q_3^* = \frac{S_{00} + S_{33}}{2} - \frac{1}{4} \text{Sp } L = \frac{S_{00} + S_{33} - S_{11} - S_{22}}{4}. \end{aligned} \tag{14}$$

Then, by means of the identities

$$\begin{aligned} S_{01} + iA_{23} &= -2q_0q_1^* , & S_{23} + iA_{01} &= 2q_2^*q_3 , \\ S_{02} + iA_{31} &= -2q_0q_2^* , & S_{31} + iA_{02} &= 2q_3^*q_1 , \\ S_{03} + iA_{12} &= -2q_0q_3^* , & S_{12} + iA_{03} &= 2q_1^*q_2 . \end{aligned} \quad (15)$$

or, using the complex polar representation,[15]

$$\begin{aligned} S_{01} + iA_{23} &= -2Q_0Q_1e^{i(\alpha_0-\alpha_1)} , & S_{23} + iA_{01} &= 2Q_2Q_3e^{i(-\alpha_2+\alpha_3)} , \\ S_{02} + iA_{31} &= -2Q_0Q_2e^{i(\alpha_0-\alpha_2)} , & S_{31} + iA_{02} &= 2Q_3Q_1e^{i(-\alpha_3+\alpha_1)} , \\ S_{03} + iA_{12} &= -2Q_0Q_3e^{i(\alpha_0-\alpha_3)} , & S_{12} + iA_{03} &= 2Q_1Q_2e^{i(-\alpha_1+\alpha_2)} . \end{aligned} \quad (16)$$

Using three equations from the above ones, we express the phases $\alpha_1, \alpha_2, \alpha_3$ through α_0 :

$$\begin{aligned} e^{i\alpha_1} &= -\frac{2Q_0Q_1}{S_{01} + iA_{23}}e^{i\alpha_0} \implies q_1 = -\frac{2Q_1^2}{S_{01} + iA_{23}}q_0 , \\ e^{i\alpha_2} &= -\frac{2Q_0Q_2}{S_{02} + iA_{31}}e^{i\alpha_0} \implies q_2 = -\frac{2Q_2^2}{S_{02} + iA_{31}}q_0 , \\ e^{i\alpha_3} &= -\frac{2Q_0Q_3}{S_{03} + iA_{12}}e^{i\alpha_0} \implies q_3 = -\frac{2Q_3^2}{S_{03} + iA_{12}}q_0 , \end{aligned} \quad (17)$$

Now, it remains to consider the basic restriction on parameters $q_0^2 - \mathbf{q}^2 = +1$; so we get

$$q_0^2 \left[1 - \frac{4Q_1^4}{(S_{01} + iA_{23})^2} - \frac{4Q_2^4}{(S_{02} + iA_{31})^2} - \frac{4Q_3^4}{(S_{03} + iA_{12})^2} \right] = +1 . \quad (18)$$

Thus, we find q_0 :

$$q_0 = \pm \left[1 - \frac{4Q_1^4}{(S_{01} + iA_{23})^2} - \frac{4Q_2^4}{(S_{02} + iA_{31})^2} - \frac{4Q_3^4}{(S_{03} + iA_{12})^2} \right]^{-1/2} . \quad (19)$$

and then arrive at expression of q_j in terms of q_0, S_{0j}, A_{ij} and Q_j ($j = \overline{1, 3}$):

$$q_1 = -\frac{2Q_1^2}{S_{01} + iA_{23}}q_0 , \quad q_2 = -\frac{2Q_2^2}{S_{02} + iA_{31}}q_0 , \quad q_3 = -\frac{2Q_3^2}{S_{03} + iA_{12}}q_0 . \quad (20)$$

II. ON IDENTIFYING LORENTZ-MUELLER MATRICES WITHIN THE TOTAL LINEAR GROUP $GL(4, R)$

The Mueller matrices of Lorentzian type $M = L$ form a real subgroup of the total linear group $GL(4, R)$

$$G = \begin{pmatrix} k_0 + \mathbf{k} \vec{\sigma} & n_0 + \mathbf{n} \vec{\sigma} \\ l_0 + \mathbf{l} \vec{\sigma} & m_0 + \mathbf{m} \vec{\sigma} \end{pmatrix} , \quad (21)$$

where $k_0 \equiv k_0 I_2$ and $\sigma_0 = I_2$, $\mathbf{k} = (k_1, k_2, k_3)$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. To any G of the form (21), there corresponds a natural invariant, its determinant (whose explicit form in such parametrization was given in [9]):

$$\det G = (kk) (mm) + (nn) (ll) - 2 (kn) (ml) - 2 (-k_0 \mathbf{n} + n_0 \mathbf{k} + i \mathbf{k} \times \mathbf{n}) (-m_0 \mathbf{l} + l_0 \mathbf{m} + i \mathbf{m} \times \mathbf{l}), \quad (22)$$

where the notation (kk) means $(kk) = k_0^2 - \mathbf{k}^2$, etc.

In [5–7], a special classification for degenerate matrices with vanishing determinant was developed; only part of them may be of Mueller type. Such a class of degenerate matrices is not considered here. Considering the above Lorentz–Mueller matrix as consisting of four blocks

$$L(q, q^*) = \begin{pmatrix} k_0 + k_3 & k_1 - ik_2 & n_0 + n_3 & n_1 - in_2 \\ k_1 + ik_2 & k_0 - k_3 & n_1 + in_2 & n_0 - n_3 \\ l_0 + l_3 & l_1 - il_2 & m_0 + m_3 & m_1 - im_2 \\ l_1 + il_2 & l_0 - l_3 & m_1 + im_2 & m_0 - m_3 \end{pmatrix} = \begin{pmatrix} K & N \\ L & M \end{pmatrix}, \quad (23)$$

we can find the associated 16 coefficients k, m, n, l by fixing an arbitrary matrix $L(q, q^*)$. We shall use the explicit form for elements of the matrix L . It is convenient to introduce the change of variables

$$k_2 \implies ik_2, \quad m_2 \implies im_2, \quad n_2 \implies in_2, \quad l_2 \implies il_2.$$

After some simple calculation, we get

$$\begin{aligned} k_0 &= q_0 q_0^* + q_1 q_1^*, & k_1 &= -(q_0 q_1^* + q_1 q_0^*), \\ k_2 &= i(q_2 q_3^* - q_3 q_2^*), & k_3 &= q_2 q_2^* + q_3 q_3^*, \\ m_0 &= q_0 q_0^* - q_1 q_1^*, & m_1 &= q_2 q_3^* + q_3 q_2^*, \\ m_2 &= i(q_0 q_1^* - q_1 q_0^*), & m_3 &= -q_2 q_2^* - q_3 q_3^*, \\ 2l_0 &= -(q_0 q_2^* + q_2 q_0^*) + i(q_0 q_2^* - q_2 q_0^*) + i(q_1 q_3^* - q_3 q_1^*) + (q_1 q_3^* + q_3 q_1^*), \\ 2l_3 &= -(q_0 q_2^* + q_2 q_0^*) - i(q_0 q_2^* - q_2 q_0^*) + i(q_1 q_3^* - q_3 q_1^*) - (q_1 q_3^* + q_3 q_1^*), \\ 2l_1 &= -i(q_0 q_3^* - q_3 q_0^*) - (q_0 q_3^* + q_3 q_0^*) + (q_1 q_2^* + q_2 q_1^*) - i(q_1 q_2^* - q_2 q_1^*), \\ 2l_2 &= -i(q_0 q_3^* - q_3 q_0^*) + (q_0 q_3^* + q_3 q_0^*) + (q_1 q_2^* + q_2 q_1^*) + i(q_1 q_2^* - q_2 q_1^*), \\ 2n_0 &= -(q_0 q_2^* + q_2 q_0^*) - i(q_0 q_2^* - q_2 q_0^*) - i(q_1 q_3^* - q_3 q_1^*) + (q_1 q_3^* + q_3 q_1^*), \\ 2n_3 &= -(q_0 q_2^* + q_2 q_0^*) + i(q_0 q_2^* - q_2 q_0^*) - i(q_1 q_3^* - q_3 q_1^*) - (q_1 q_3^* + q_3 q_1^*), \\ 2n_1 &= -(q_0 q_3^* + q_3 q_0^*) + i(q_0 q_3^* - q_3 q_0^*) + i(q_1 q_2^* - q_2 q_1^*) + (q_1 q_2^* + q_2 q_1^*), \\ 2n_2 &= -(q_0 q_3^* + q_3 q_0^*) - i(q_0 q_3^* - q_3 q_0^*) + i(q_1 q_2^* - q_2 q_1^*) - (q_1 q_2^* + q_2 q_1^*). \end{aligned} \quad (24)$$

III. ON THE EXPANSION OF THE LORENTZ MATRICES IN DIRAC BASIS

We can alternatively rephrase the previous problem: one can decompose the Lorentz matrix in terms of the 16 Dirac matrices

$$L(q, q^*) = Z + \gamma^5 \tilde{Z} + \gamma^l Z_l + \gamma^l \gamma^5 \tilde{Z}_l + \sigma^{mn} Z_{mn} ; \quad (25)$$

where the 16 coefficients are given by the formulas[16]

$$\begin{aligned} Z &= \frac{1}{4} \text{Sp } L(q, q^*) , & \tilde{Z} &= \frac{1}{4} \text{Sp } \gamma^5 L(q, q^*) , \\ Z_k &= \frac{1}{4} \text{Sp } \gamma_k L(q, q^*) , & \tilde{Z}_k &= \frac{1}{4} \text{Sp } \gamma^5 \gamma_k L(q, q^*) , & Z_{kl} &= -\frac{1}{2} \text{Sp } \sigma_{kl} L(q, q^*) . \end{aligned} \quad (26)$$

After simple calculations we arrive at the formulas

$$\begin{aligned} Z &= q_0 q_0^* , & \tilde{Z} &= q_1 q_1^* , & Z_{03} &= q_3 q_3^* , & -iZ_{12} &= q_2 q_2^* , \\ Z_{01} &= -\frac{1}{2} [(q_0 q_1^* + q_1 q_0^*) + (q_2 q_3^* + q_3 q_2^*)] , & Z_{23} &= \frac{i}{2} [-(q_0 q_1^* + q_1 q_0^*) + (q_2 q_3^* + q_3 q_2^*)] , \\ Z_{02} &= \frac{1}{2} [(q_0 q_1^* - q_1 q_0^*) - (q_2 q_3^* - q_3 q_2^*)] , & Z_{31} &= \frac{i}{2} [-(q_0 q_1^* - q_1 q_0^*) - (q_2 q_3^* - q_3 q_2^*)] , \\ Z_0 &= \frac{1}{2} [-(q_0 q_2^* + q_2 q_0^*) + (q_1 q_3^* + q_3 q_1^*)] , & \tilde{Z}_0 &= \frac{i}{2} [(q_0 q_2^* - q_2 q_0^*) + (q_1 q_3^* - q_3 q_1^*)] , \\ Z_3 &= -\frac{i}{2} [(q_0 q_2^* - q_2 q_0^*) - (q_1 q_3^* - q_3 q_1^*)] , & \tilde{Z}_3 &= -\frac{1}{2} [(q_0 q_2^* + q_2 q_0^*) + (q_1 q_3^* + q_3 q_1^*)] , \\ Z_1 &= -\frac{i}{2} (q_0 q_3^* - q_3 q_0^* + q_1 q_2^* - q_2 q_1^*) , & \tilde{Z}_1 &= -\frac{1}{2} [(q_0 q_3^* + q_3 q_0^*) - (q_1 q_2^* + q_2 q_1^*)] , \\ Z_2 &= -\frac{i}{2} (-q_0 q_3^* - q_3 q_0^* - q_1 q_2^* - q_2 q_1^*) , & \tilde{Z}_2 &= -\frac{1}{2} [-(q_0 q_3^* - q_3 q_0^*) + (q_1 q_2^* - q_2 q_1^*)] . \end{aligned} \quad (27)$$

On the ground of the formulas (27), one can develop a method to find the parameters q_a by means of the coefficients in (25). Indeed, starting with (27), we note

$$Z = q_0 q_0^* , \quad \tilde{Z} = q_1 q_1^* , \quad Z_{03} = q_3 q_3^* , \quad -iZ_{12} = q_2 q_2^* ; \quad (28)$$

then

$$\begin{aligned} Z_{01} + iZ_{23} &= -q_2 q_3^* - q_2^* q_3 , & Z_{01} - iZ_{23} &= -q_0 q_1^* - q_0^* q_1 , \\ Z_{02} + iZ_{31} &= q_0 q_1^* - q_0^* q_1 , & Z_{02} - iZ_{31} &= -q_2 q_3^* + q_2^* q_3 , \end{aligned}$$

that is

$$\begin{aligned} (Z_{01} - iZ_{23}) + (Z_{02} + iZ_{31}) &= -2 q_0^* q_1 , \\ (Z_{01} - iZ_{23}) - (Z_{02} + iZ_{31}) &= -2 q_0 q_1^* , \\ (Z_{01} + iZ_{23}) + (Z_{02} - iZ_{31}) &= -2 q_2 q_3^* , \\ (Z_{01} + iZ_{23}) - (Z_{02} - iZ_{31}) &= -2 q_2^* q_3 ; \end{aligned} \quad (29)$$

then

$$\begin{aligned} Z_0 - \tilde{Z}_3 &= (q_1 q_3^* + q_1^* q_3) , & Z_0 + \tilde{Z}_3 &= -(q_0 q_2^* + q_0^* q_2) , \\ \tilde{Z}_0 - Z_3 &= i(q_0 q_2^* - q_0^* q_2) , & \tilde{Z}_0 + Z_3 &= i(q_1 q_3^* - q_1^* q_3) , \end{aligned}$$

that is

$$\begin{aligned} (Z_0 - \tilde{Z}_3) + i(\tilde{Z}_0 + Z_3) &= 2 q_1^* q_3 , \\ (Z_0 - \tilde{Z}_3) - i(\tilde{Z}_0 + Z_3) &= 2 q_1 q_3^* ; \\ (Z_0 + \tilde{Z}_3) + i(\tilde{Z}_0 - Z_3) &= -2 q_0 q_2^* , \\ (Z_0 + \tilde{Z}_3) - i(\tilde{Z}_0 - Z_3) &= -2 q_0^* q_2 ; \end{aligned} \tag{30}$$

and then

$$\begin{aligned} Z_1 + i\tilde{Z}_2 &= -i(q_1 q_2^* - q_1^* q_2) , & Z_1 - i\tilde{Z}_2 &= -i(q_0 q_3^* - q_0^* q_3) , \\ \tilde{Z}_1 + iZ_2 &= q_0 q_3^* - q_0^* q_3 , & \tilde{Z}_1 - iZ_2 &= q_1 q_2^* + q_1^* q_2 . \end{aligned}$$

that is

$$\begin{aligned} (Z_1 + i\tilde{Z}_2) + i(\tilde{Z}_1 - iZ_2) &= +2i q_1^* q_2 , \\ (Z_1 + i\tilde{Z}_2) - i(\tilde{Z}_1 - iZ_2) &= -2i q_1 q_2^* , \\ (Z_1 - i\tilde{Z}_2) + i(\tilde{Z}_1 + iZ_2) &= -2i q_0 q_3^* , \\ (Z_1 - i\tilde{Z}_2) - i(\tilde{Z}_1 + iZ_2) &= +2i q_0^* q_3 . \end{aligned} \tag{31}$$

The previously produced formulas allow us to calculate the parameters for Lorentz matrices. Indeed, let us take relations

$$\begin{aligned} Z &= q_0 q_0^* & q_0 &= \sqrt{Z} e^{i\alpha} , \\ q_1 &= -\frac{1}{2 q_0^*} [(Z_{01} - iZ_{23}) + (Z_{02} + iZ_{31})] = \frac{1}{q_0^*} M_1 , \\ q_2 &= -\frac{1}{2 q_0^*} [(Z_0 + \tilde{Z}_3) - i(\tilde{Z}_0 - Z_3)] = \frac{1}{q_0^*} M_2 , \\ q_3 &= -i\frac{1}{2 q_0^*} [(Z_1 - i\tilde{Z}_2) - i(\tilde{Z}_1 + iZ_2)] = \frac{1}{q_0^*} M_3 . \end{aligned} \tag{32}$$

With the us of additional quadratic restriction $q_0^2 - \mathbf{q}^2 = 1$, we derive the formulas

$$e^{i\alpha} = \pm \sqrt{\frac{Z}{Z^2 - \mathbf{M}^2}} , \quad q_0 = \sqrt{Z} e^{i\alpha} , \quad q_j = \frac{e^{i\alpha}}{\sqrt{Z}} M_j . \tag{33}$$

The expansion of the Lorentz matrices in the Dirac basis can be written as

$$L = \begin{pmatrix} Z + \tilde{Z} + \Sigma^{mn} Z_{mn} & \bar{\sigma}^n (Z_n - \tilde{Z}_n) \\ \sigma^n (Z_n + \tilde{Z}_n) & Z - \tilde{Z} + \bar{\Sigma}^{mn} Z_{mn} \end{pmatrix} \quad (34)$$

where

$$\begin{aligned} \sigma^a &= (I, +\sigma^j), & \sigma_a &= (I, -\sigma^j), \\ \gamma^a &= \begin{pmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, & \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix} &= \gamma^5, & \sigma^{ab} &= \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) = \begin{pmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{pmatrix}, \\ \Sigma^{01} &= \frac{1}{4}(\bar{\sigma}^0 \sigma^1 - \bar{\sigma}^1 \sigma^0) = \frac{1}{2}\sigma^1, & \Sigma^{02} &= \frac{1}{2}\sigma^2, & \Sigma^{03} &= \frac{1}{2}\sigma^3, \\ \Sigma^{12} &= \frac{1}{4}(\bar{\sigma}^1 \sigma^2 - \bar{\sigma}^2 \sigma^1) = -\frac{i}{2}\sigma^3, & \Sigma^{23} &= -\frac{i}{2}\sigma^1, & \Sigma^{31} &= -\frac{i}{2}\sigma^2; \\ \bar{\Sigma}^{01} &= \frac{1}{4}(\sigma^0 \bar{\sigma}^1 - \sigma^1 \bar{\sigma}^0) = -\frac{1}{2}\sigma^1, & \bar{\Sigma}^{02} &= -\frac{1}{2}\sigma^2, & \bar{\Sigma}^{03} &= -\frac{1}{2}\sigma^3, \\ \bar{\Sigma}^{12} &= \frac{1}{4}(\sigma^1 \bar{\sigma}^2 - \sigma^2 \bar{\sigma}^1) = -\frac{i}{2}\sigma^3, & \bar{\Sigma}^{23} &= -\frac{i}{2}\sigma^1, & \bar{\Sigma}^{31} &= -\frac{i}{2}\sigma^2. \end{aligned} \quad (35)$$

Relation (34) reduces to the form

$$L = \begin{pmatrix} Z + \tilde{Z} + \sigma^1 Z_1^- + \sigma^2 Z_2^- + \sigma^3 Z_3^- & Z_0 - \tilde{Z}_0 - \sigma^j (Z_j - \tilde{Z}_j) \\ Z_0 + \tilde{Z}_0 + \sigma^j (Z_j + \tilde{Z}_j) & Z - \tilde{Z} - \sigma^1 Z_1^+ - \sigma^2 Z_2^+ - \sigma^3 Z_3^+ \end{pmatrix}, \quad (36)$$

where

$$\begin{aligned} Z_1^- &= +Z_{01} - iZ_{23}, & Z_2^- &= +Z_{02} - iZ_{31}, & Z_3^- &= +Z_{03} - iZ_{12}, \\ Z_1^+ &= Z_{01} + iZ_{23}, & Z_2^+ &= Z_{02} + iZ_{31}, & Z_3^+ &= Z_{03} + iZ_{12}. \end{aligned} \quad (37)$$

The matrix (36) can be considered as consisting of four blocks

$$L = \begin{pmatrix} k_0 + k_j \sigma^j & n_0 + n_j \sigma^j \\ l_0 + l_j \sigma^j & m_0 + m_j \sigma^j \end{pmatrix}; \quad (38)$$

and explicit expressions for the parameters are

$$\begin{aligned} k_0 &= Z + \tilde{Z}, & m_0 &= Z - \tilde{Z}, & k_j &= +Z_j^-, & m_j &= -Z_j^+, \\ n_0 &= Z_0 - \tilde{Z}_0, & l_0 &= Z_0 + \tilde{Z}_0, & n_j &= -Z_j + \tilde{Z}_j, & l_j &= Z_j + \tilde{Z}_j; \end{aligned} \quad (39)$$

and it can be readily verified that these coincide with (23).

IV. ON PARAMETERS OF LORENTZ–MUELLER MATRICES AND TRANSITIVITY RELATIONS

Let us start with the factorized form of Lorentz transformations

$$\begin{aligned} AA^* &= L & \implies & A = L(A^*)^{-1}, \\ A^*A &= L & \implies & A^* = LA^{-1}. \end{aligned} \quad (40)$$

whence we get

$$L[(A^*)^{-1} + A^{-1}] = A + A^*, \quad L[(A^*)^{-1} - A^{-1}] = A - A^*, \quad (41)$$

where

$$\begin{aligned} A &= \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ -q_1 & q_0 & -iq_3 & iq_2 \\ -q_2 & iq_3 & q_0 & -iq_1 \\ -q_3 & -iq_2 & iq_1 & q_0 \end{pmatrix}, & A^* &= \begin{pmatrix} q_0^* & -q_1^* & -q_2^* & -q_3^* \\ -q_1^* & q_0^* & iq_3^* & -iq_2^* \\ -q_2^* & -iq_3^* & q_0^* & iq_1^* \\ -q_3^* & iq_2^* & -iq_1^* & q_0^* \end{pmatrix}, \\ A^{-1} &= \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ q_1 & q_0 & iq_3 & -iq_2 \\ q_2 & -iq_3 & q_0 & iq_1 \\ q_3 & iq_2 & -iq_1 & q_0 \end{pmatrix}, & (A^*)^{-1} &= \begin{pmatrix} q_0^* & q_1^* & q_2^* & q_3^* \\ q_1^* & q_0^* & -iq_3^* & iq_2^* \\ q_2^* & iq_3^* & q_0^* & -iq_1^* \\ q_3^* & -iq_2^* & iq_1^* & q_0^* \end{pmatrix}. \end{aligned}$$

For the special case of Euclidean rotation $q_0 = n_0$, $q_j = -in_j$, the relations (41) give

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & L_{11} & L_{12} & L_{13} \\ 0 & L_{21} & L_{22} & L_{23} \\ 0 & L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} n_0 & 0 & 0 & 0 \\ 0 & n_0 & n_3 & -n_2 \\ 0 & -n_3 & n_0 & n_1 \\ 0 & n_2 & -n_1 & n_0 \end{pmatrix} &= \begin{pmatrix} n_0 & 0 & 0 & 0 \\ 0 & n_0 & -n_3 & n_2 \\ 0 & n_3 & n_0 & -n_1 \\ 0 & -n_2 & n_1 & n_0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & L_{11} & L_{12} & L_{13} \\ 0 & L_{21} & L_{22} & L_{23} \\ 0 & L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} 0 & in_1 & in_2 & in_3 \\ in_1 & 0 & 0 & 0 \\ in_2 & 0 & 0 & 0 \\ in_3 & 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & in_1 & in_2 & in_3 \\ in_1 & 0 & 0 & 0 \\ in_2 & 0 & 0 & 0 \\ in_3 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

In fact, we have four transitivity relations

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} n_0 \\ -n_3 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_0 \\ n_3 \\ -n_2 \end{pmatrix},$$

$$\begin{aligned}
\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} n_3 \\ n_0 \\ -n_1 \end{pmatrix} &= \begin{pmatrix} -n_3 \\ n_0 \\ n_1 \end{pmatrix}, \\
\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} -n_2 \\ n_1 \\ n_0 \end{pmatrix} &= \begin{pmatrix} n_2 \\ -n_1 \\ n_0 \end{pmatrix}, \\
\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} &= \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix},
\end{aligned} \tag{42}$$

where

$$O = \begin{pmatrix} 1 - 2(n_2^2 + n_3^2) & -2n_0n_3 + 2n_1n_2 & +2n_0n_2 + 2n_1n_3 \\ +2n_0n_3 + 2n_1n_2 & 1 - 2(n_3^2 + n_1^2) & -2n_0n_1 + 2n_2n_3 \\ -2n_0n_2 + 2n_1n_3 & +2n_0n_1 + 2n_2n_3 & 1 - 2(n_1^2 + n_2^2) \end{pmatrix}, \quad n_0^2 + n_1^2 + n_2^2 + n_3^2 = 1. \tag{43}$$

It can be readily verified that these four relations are valid indeed. In the context of the problems of polarization optics, the most interesting is the last one from (42)

$$O \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \tag{44}$$

However, all the four relations allow physical interpretation in the frames of polarization optics: they provide us with simple transitivity relations that describe the action of an optical element on a specially chosen probe of light beams.

In a similar manner, one may consider the case of pseudo-Euclidean rotations $q_0 = m_0$, $q_j = m_j$, whence (41) take the form

$$\begin{aligned}
\begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} m_0 & m_1 & m_2 & m_3 \\ m_1 & m_0 & 0 & 0 \\ m_2 & 0 & m_0 & 0 \\ m_3 & 0 & 0 & m_0 \end{pmatrix} &= \begin{pmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & m_0 & 0 & 0 \\ -m_2 & 0 & m_0 & 0 \\ -m_3 & 0 & 0 & m_0 \end{pmatrix}, \\
\begin{pmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -im_3 & im_2 \\ 0 & im_3 & 0 & -im_1 \\ 0 & -im_2 & im_1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -im_3 & im_2 \\ 0 & im_3 & 0 & -im_1 \\ 0 & -im_2 & im_1 & 0 \end{pmatrix}.
\end{aligned} \tag{45}$$

Using the explicit formulas

$$m_0 = \text{ch } \chi, \quad m_j = \text{sh } \chi e_j,$$

where $\mathbf{e}^2 = 1$ which infer $m_0^2 - \mathbf{m}^2 = 1$, we get

$$L = \begin{pmatrix} m_0^2 + \mathbf{m}^2 & -2m_0m_1 & -2m_0m_2 & -2m_0m_3 \\ -2m_0m_1 & 1 + 2m_1^2 & 2m_1m_2 & 2m_1m_3 \\ -2m_0m_2 & 2m_1m_2 & 1 + 2m_2^2 & 2m_2m_3 \\ -2m_0m_3 & 2m_3m_1 & 2m_2m_3 & 1 + 2m_3^2 \end{pmatrix}, \quad (46)$$

and one can easily verify that such an equation holds indeed

$$L \begin{pmatrix} m_0 & m_1 & m_2 & m_3 \\ m_1 & m_0 & 0 & 0 \\ m_2 & 0 & m_0 & 0 \\ m_3 & 0 & 0 & m_0 \end{pmatrix} = \begin{pmatrix} m_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & m_0 & 0 & 0 \\ -m_2 & 0 & m_0 & 0 \\ -m_3 & 0 & 0 & m_0 \end{pmatrix},$$

$$L \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -im_3 & im_2 \\ 0 & im_3 & 0 & -im_1 \\ 0 & -im_2 & im_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -im_3 & im_2 \\ 0 & im_3 & 0 & -im_1 \\ 0 & -im_2 & im_1 & 0 \end{pmatrix}. \quad (47)$$

Thus, there arise 7 non-trivial transitivity relations. Among the 7 vectors from (47), only one (the first) is time-similar

$$m_0^2 - m_1^2 - m_2^2 - m_3^2 = 1, \quad (48)$$

and the remaining ones are space-similar; for instance, $m_1^2 - m_0^2 = -(1 + m_2^2 + m_3^2) < 0$. In the context of polarization optics, only this 4-vector is of interest and can be considered as representing the Stokes 4-vector.

$$L \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m_0 \\ -m_1 \\ -m_2 \\ -m_3 \end{pmatrix}; \quad (49)$$

and we may compare it with (44).

Now, let us turn to general case of arbitrary Mueller matrices of Lorentz type

$$L [(A^*)^{-1} + A^{-1}] = A + A^*, \quad L [(A^*)^{-1} - A^{-1}] = A - A^*; \quad (50)$$

Re-expressing the parameters (q_0, q_1, q_2, q_3) in terms of their real and imaginary parts, we have

$$\begin{aligned} q_0 &= x_0 + iy_0, & q_j &= x_j + iy_j, & q_0^* &= x_0 - iy_0, & q_j^* &= x_j - iy_j, \\ x_0^2 - x_1^2 - x_2^2 - x_3^2 - y_0^2 + y_1^2 + y_2^2 + y_3^2 &= 1, & x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 &= 0. \end{aligned} \quad (51)$$

The relations (50) become equivalent to the following

$$\begin{aligned} L \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -y_3 & y_2 \\ x_2 & y_3 & x_0 & -y_1 \\ x_3 & -y_2 & y_1 & x_0 \end{pmatrix} &= \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ -x_1 & x_0 & y_3 & -y_2 \\ -x_2 & -y_3 & x_0 & y_1 \\ -x_3 & y_2 & -y_1 & x_0 \end{pmatrix}, \\ L \begin{pmatrix} -y_0 & -y_1 & -y_2 & -y_3 \\ -y_1 & -y_0 & -x_3 & x_2 \\ -y_2 & x_3 & -y_0 & -x_1 \\ -y_3 & -x_2 & x_1 & -y_0 \end{pmatrix} &= \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ -y_1 & y_0 & -x_3 & x_2 \\ -y_2 & x_3 & y_0 & -x_1 \\ -y_3 & -x_2 & x_1 & y_0 \end{pmatrix}. \end{aligned} \quad (52)$$

Explicitly, the corresponding transitivity relations (50) read

$$\begin{aligned} \psi_0 &= \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad L\psi_0 = \psi'_0 = \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} x_1 \\ x_0 \\ y_3 \\ -y_2 \end{pmatrix}, \quad L\psi_1 = \psi'_1 = \begin{pmatrix} -x_1 \\ x_0 \\ -y_3 \\ y_2 \end{pmatrix}, \\ \Psi_2 &= \begin{pmatrix} x_2 \\ -y_3 \\ x_0 \\ y_1 \end{pmatrix}, \quad L\Psi_2 = \Psi'_2 = \begin{pmatrix} -x_2 \\ y_3 \\ x_0 \\ -y_1 \end{pmatrix}, \quad \Psi_3 = \begin{pmatrix} x_3 \\ y_2 \\ -y_1 \\ x_0 \end{pmatrix}, \quad L\Psi_3 = \Psi'_3 = \begin{pmatrix} -x_3 \\ -y_2 \\ y_1 \\ x_0 \end{pmatrix}, \\ L\Phi_0 &= \Phi'_0 = \begin{pmatrix} y_0 \\ -y_1 \\ -y_2 \\ -y_3 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} -y_1 \\ -y_0 \\ x_3 \\ -x_2 \end{pmatrix}, \quad L\Phi_1 = \Phi'_1 = \begin{pmatrix} -y_1 \\ y_0 \\ x_3 \\ -x_2 \end{pmatrix}, \\ \Phi_2 &= \begin{pmatrix} -y_2 \\ -x_3 \\ -y_0 \\ x_1 \end{pmatrix}, \quad L\Phi_2 = \Phi'_2 = \begin{pmatrix} -y_2 \\ -x_3 \\ y_0 \\ x_1 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} -y_3 \\ x_2 \\ -x_1 \\ -y_0 \end{pmatrix}, \quad L\Phi_3 = \Phi'_3 = \begin{pmatrix} -y_3 \\ x_2 \\ -x_1 \\ y_0 \end{pmatrix}. \end{aligned} \quad (53)$$

It should be noted that the four cases describe transformations over 4-vectors which change the sign of the zero component into the inverse one, therefore the corresponding 4-vectors are space-similar

ones and these cannot describe Stokes 4-vectors. Indeed, for an arbitrary time-similar vector, we have

$$I. \quad t^2 > x^2, \quad t > x, \quad t' = \frac{e^\beta + e^{-\beta}}{2}t - \frac{e^\beta - e^{-\beta}}{2}x > 0 ;$$

where instead, for a space-similar vector, we have

$$II. \quad t^2 < x^2, \quad \underline{t < x}, \quad t' = \frac{e^\beta + e^{-\beta}}{2}t - \frac{e^\beta - e^{-\beta}}{2}x \implies$$

$$\text{possible } t' < 0, \quad \text{if} \quad \frac{e^\beta + e^{-\beta}}{2}t < \frac{e^\beta - e^{-\beta}}{2}x, \quad \frac{e^{2\beta} + 1}{e^{2\beta} - 1} < \frac{x}{t}.$$

As Stokes 4-vectors one can consider only the four vectors from the above: $\Psi_0, \Phi_1, \Phi_2, \Phi_3$.

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- [11] Hereafter, we denote by "*" the complex conjugation.
- [12] We denote $q = (q_0, \mathbf{q}) \equiv (q_0, q_1, q_2, q_3)$, and recall that we further assume that $q_0^2 - \mathbf{q}^2 = +1$, where $\mathbf{q}^2 = \langle \mathbf{q}, \mathbf{q} \rangle$.
- [13] We further denote by dash the transposition of matrices.
- [14] Cf. [3], $L = S + A$, with $S = S^t$, $A = -A^t$.
- [15] Here, we denote $Q_k = \text{abs}(q_k)$, and $\alpha_k = \text{arg}(q_k)$.
- [16] We denote by Sp the trace of the corresponding matrix.

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